# Notes for ‘An overview of Springer Theory’

Iordan Ganev

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**Abstract**

These notes were prepared for a talk given by the author at the graduate student geometry seminar at the University of Texas at Austin. The aim of the talk is to demonstrate how to produce convolution algebras associated to fiber products of certain maps and how this geometric construction gives rise to classification results in representation theory. We present three contexts for this philosophy: functions on finite sets, the Springer correspondence for Weyl groups, and Springer theory for $\mathfrak{sl}_n$.

## 1 Introduction

A standard problem in representation theory is to classify the simple modules of a given algebra. In this talk, we explore examples in which geometric constructions make such classifications possible. The basic pattern is as follows:

- Let $\mu : X \to Y$ be a map of varieties or finite sets.
- Use the fiber product $Z = X \times_Y X$ to produce a convolution algebra $A$.
- The algebra $A$ acts on the fibers of $\mu$ (in the appropriate sense).
- Analysis of these actions leads to a classification of simple $A$-modules.

Some of the relevant techniques can be witnessed in the elementary setting of functions on finite sets. A classical instance of the above pattern is the Springer correspondence for irreducible representations of the Weyl group of a semisimple complex Lie algebra. To state the result, we introduce the Springer resolution and briefly discuss Borel-Moore homology. The next example is a classification for finite-dimensional irreducible representations of $\mathfrak{sl}_n$. The final portion of the talk will consist of remarks on affine Hecke algebras and connections to the Langlands program.

## 2 Functions on finite sets

In this section we present a convolution map for functions on finite sets. This construction will lead to the notion of a convolution algebra and also to a classification theorem for its modules. The manipulations and arguments are elementary; nonetheless, when stated correctly, they hold verbatim in more sophisticated settings. The reader is invited to treat any unfilled details in this section as exercises.

## 2.1 Composition and convolution

For a finite set $X$, let $\mathbb{C}[X]$ denote the vector space of complex-valued functions on $X$. Let $X_1$, $X_2$, and $X_3$ be finite sets and consider the diagram

$$\begin{tikzcd} & X_1 \times X_2 \times X_3 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X_1 \times X_2 & X_1 \times X_3 & X_2 \times X_3 \end{tikzcd}$$

For subsets $Z_{12} \subset X_1 \times X_2$ and $Z_{23} \subset X_2 \times X_3$, we define their set-theoretic 'composition' as

$$Z_{12} \circ Z_{23} = \pi_{13}(\pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(Z_{23})) \subset X_1 \times X_3.$$

In words, first take the inverse image of each of the sets in $X_1 \times X_2 \times X_3$, then intersect those inverse images, and finally take the image of the intersection under the projection to $X_1 \times X_3$. This is our first appearance of the fundamental pattern of pulling back, intersecting, and pushing forward.

**Exercise.** If $Z_{12} = \text{graph}(X_1 \overset{\alpha}{\to} X_2)$ and $Z_{23} = \text{graph}(X_2 \overset{\beta}{\to} X_3)$ are each the graph of a function, show that $Z_{12} \circ Z_{23}$ is the graph of the composition $\beta \circ \alpha$.

The set-theoretic composition leads to a linear 'convolution' map on the level of functions, defined by pulling back, multiplying pointwise, and pushing forward:

$$\begin{aligned} \mathbb{C}[Z_{12}] \times \mathbb{C}[Z_{23}] &\to \mathbb{C}[Z_{12} \circ Z_{23}] \\ f, g &\mapsto \pi_{13*}(\pi_{12}^* f \cdot \pi_{23}^* g) \end{aligned}$$

Here the pullback $\pi_{12}^* f$ is defined on $a \in X_1 \times X_2 \times X_3$ as the composition $f \circ \pi_{12}(a)$ if $\pi_{12}(a) \in Z_{12}$ and zero otherwise. The same remark holds for $\pi_{23}^* g$. Pushforward is defined using summation over fibers, and convolution admits the following explicit formula

$$(f * g)(x_1, x_3) = \sum_{x_2 \in S} f(x_1, x_2) \cdot g(x_2, x_3)$$

where $S = \{x_2 \in X_2 \mid (x_1, x_2) \in Z_{12}, (x_2, x_3) \in Z_{23}\}$.

This construction is associative in the following sense. Let $X_4$ be a fourth finite set and let $Z_{34} \subset X_3 \times X_4$. One first checks that set-theoretic composition is associative: $(Z_{12} \circ Z_{23}) \circ Z_{34} = Z_{12} \circ (Z_{23} \circ Z_{34})$ as subsets of $X_1 \times X_4$. Then one checks that if $f_{ij} \in \mathbb{C}[Z_{ij}]$, then

$$(f_{12} * f_{23}) * f_{34} = f_{12} * (f_{23} * f_{34})$$

as elements of $\mathbb{C}[Z_{12} \circ Z_{23} \circ Z_{34}]$. This last verification can be performed using explicit formulas, or by manipulating pullbacks and pushforwards. For the second approach, some useful facts are (1) pullback commutes with pointwise multiplication and (2) the projection formula $\alpha_*(\alpha^* f \cdot g) = f \cdot \alpha_* g$ for any function $\alpha : X \to Y$ and any $f \in \mathbb{C}[Y]$, $g \in \mathbb{C}[X]$.

### 2.2 Convolution algebras

Important instances of convolution occur in the context of a map $\mu : X \to Y$ of finite sets. The fiber product of $X$ with itself over $Y$ is defined as

$$Z = X \times_Y X = \{(x, x') \in X \times X \mid \mu(x) = \mu(x')\}.$$

First, set $X_1 = X_2 = X_3 = X$ and $Z_{12} = Z_{23} = Z$. Then $Z \circ Z = Z$ and we get an algebra structure on $\mathbb{C}[Z]$:

$$* : \mathbb{C}[Z] \times \mathbb{C}[Z] \to \mathbb{C}[Z].$$

the unit is the characteristic function of the diagonal $X_\Delta$ in $Z = X \times_Y X$. An algebra that arises in this way is often called a **convolution algebra**. In the special case that $Y$ is a point, we have that $Z = X \times X$ and one can verify that the resulting convolution algebra is a matrix algebra

$$(\mathbb{C}[X \times X], *) \simeq \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X]).$$

Next, consider the diagram

$$\begin{tikzcd}
 & X \times X \times \mathrm{pt} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
X \times X & X & X
\end{tikzcd}$$

Let $Z_{12} = Z = X \times_Y X$ and take $Z_{23} = \mu^{-1}(y) = X_y$ to be the fiber over a point $y \in Y$. Then $Z \circ X_y = X_y$ and the convolution map induces a $\mathbb{C}[Z]$-module structure on $\mathbb{C}[X_y]$:

$$* : \mathbb{C}[Z] \times \mathbb{C}[X_y] \to \mathbb{C}[X_y].$$

In fact, we have:

**Proposition 1.** *The collection $\{\mathbb{C}[X_y] \mid y \in Y,\ X_y \neq \emptyset\}$ forms a complete set of isomorphism classes of irreducible modules for the convolution algebra $\mathbb{C}[Z]$.*

*Proof.* Observe that $Z = \coprod_{y \in Y} X_y \times X_y$ and there are isomorphisms of algebras

$$\mathbb{C}[Z] \simeq \bigoplus_{y \in Y} \mathbb{C}[X_y \times X_y] \simeq \bigoplus_{y \in Y} \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X_y]).$$

Since our algebra decomposes into a direct sum of matrix algebras, the proposition follows by standard results (e.g. the Wedderburn theorem). $\square$

# 3 The Springer Correspondence for Weyl Groups

### 3.1 The Nilpotent Cone and the Springer Resolution

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$. Let $\mathcal{B}$ be the set of Borel subalgebras of $\mathfrak{g}$, known as the **flag variety**. If $G$ is a Lie group with Lie algebra $\mathfrak{g}$ and $B$ is a Borel subgroup of $G$, then we can identify $\mathcal{B}$ with the homogeneous space $G/B$. This identification can be used to show that $\mathcal{B}$ is a smooth projective variety. Recall that associated to $\mathfrak{g}$ is a finite Coxeter group $W$, known as the **Weyl group** of $\mathfrak{g}$.

Let $\mathcal{N}$ be the set of elements $x \in \mathfrak{g}$ such that $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is a nilpotent linear transformation. One observes that the subvariety $\mathcal{N}$ of $\mathfrak{g}$ is stable under the adjoint action of $G$ and also under the action of $\mathbb{C}^*$ by dilations. We call $\mathcal{N}$ the **nilpotent cone** of $\mathfrak{g}$.

Note that $\mathcal{N}$ is always singular at the origin. One can define a resolution of singularities of $\mathcal{N}$ as follows. Let

$$\tilde{\mathcal{N}} = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

The group $G$ acts on $\tilde{\mathcal{N}}$ by the adjoint representation in both coordinates. The natural projection $\tilde{\mathcal{N}} \to \mathcal{B}$ identifies $\tilde{\mathcal{N}}$ with the cotangent bundle $T^*\mathcal{B}$ of the flag variety. Hence, $\tilde{\mathcal{N}}$ is a smooth variety. There is also a natural map

$$\mu : \tilde{\mathcal{N}} \to \mathcal{N}$$

defined by $\mu(x, \mathfrak{b}) = x$. One can show that this map is proper, surjective, and $G$-equivariant, and moreover, that $\#\mu^{-1}(x) = 1$ if $x$ is a regular element of $\mathcal{N}$. Since the set of regular nilpotent elements forms a Zariski-open and dense subset of $\mathcal{N}$, we see that $\mu$ is generically one to one. Finally, since $\tilde{\mathcal{N}}$ is smooth; we conclude that $\mu$ is a resolution of singularities of $\mathcal{N}$.

**Definition.** The map $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is called the **Springer resolution**. The fiber

$$\mu^{-1}(x) = \{\mathfrak{b} \in \mathcal{B} \mid x \in \mathfrak{b}\}$$

over an element $x \in \mathcal{N}$ is called a **Springer fiber** and is denoted by $\mathcal{B}_x$. The fiber product

$$Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$$

is called the **Steinberg variety**.

## 3.2 Borel-Moore homology

If we were dealing with finite sets, then we have seen that the space of functions on $Z$ carries a natural convolution algebra structure. However, the construction breaks down in the present setting of varieties. One problem is that defining the pushforward of a function requires a choice of measure. To avoid measures, one can think about differential forms, where the pointwise multiplication of functions is replaced by the wedge product of differential forms. However, certain properness assumptions are necessary, and these lead to degeneracies in the cases we are interested in. A way to avoid the properness assumptions is to use the notion of 'currents', which are (roughly speaking) distribution-like differential forms. However, the wedge product is not well-defined for currents.

Borel-Moore homology provides a solution to these difficulties. For the purposes of this talk, we will not define Borel-Moore homology precisely and kindly ask the audience and readers to take it as a black box. Borel-Moore homology is defined for any variety and has many nice properties, for example: proper pushforward, smooth pullback, intersection pairing, a Künneth formula, and fundamental classes. One can think of Borel-Moore homology as the dual notion to cohomology with compact support.

For a variety $X$, let $H_*(X)$ denote its Borel-Moore homology with complex coefficients. We will often be interested only in the subspace spanned by the fundamental classes of the irreducible components of $X$, rather than all of $H_*(X)$. We denote this subspace by $H(X)$. If all irreducible components have the same dimension, then $H(X)$ will coincide with the top degree homology $H_{\text{top}}(X)$ of $X$.

### 3.3 The Springer correspondence

The relevance of Borel-Moore homology for this talk is that it has enough features to define a convolution algebra structure on $H_*(Z)$, where $Z$ is the Steinberg variety. The procedure for defining this convolution follows the same as pattern as in the setting of finite sets. We give the fundamental diagram for the present setting, as well as the steps in defining convolution:

$$\begin{tikzcd} & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ Z \subset \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \supset Z \end{tikzcd}$$

- Pullback $c, c' \in H_*(Z)$ to classes $\pi_{12}^* c \in H_*(\pi_{12}^{-1}(Z))$ and $\pi_{23}^* c' \in H_*(\pi_{23}^{-1}(Z))$.
- Use an intersection pairing, which is the analogue to pointwise multiplication, in the ambient space $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$ to produce a class $\pi_{12}^* c \cap \pi_{23}^* c' \in H_*(\pi_{12}^{-1}(Z) \cap \pi_{23}^{-1}(Z))$.
- Finally, pushforward via $\pi_{13}$ to obtain a class in $H_*(Z)$. We denote this class by $c * c'$.

The unit of the algebra is the fundamental class of the diagonal $\tilde{\mathcal{N}}_\Delta$ in $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Dimension considerations (that we omit) imply that $H(Z)$ is a subalgebra.

**Theorem 2.** *The algebra $H(Z)$ is isomorphic to the group algebra of the Weyl group:*

$$H(Z) \simeq \mathbb{C}[W].$$

The proof (see section 3.4 of [1]) uses the facts that $Z$ is a subvariety of $\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \simeq T^*(\mathcal{B} \times \mathcal{B})$ and that the orbits of the diagonal action of $G$ on $\mathcal{B} \times \mathcal{B}$ are in bijection with the Weyl group, by the Bruhat decomposition:

$$G \backslash (\mathcal{B} \times \mathcal{B}) = G \backslash (G/B \times G/B) = B \backslash G/B \simeq W.$$

As may be expected, there is an action of $H_*(Z)$ on the homology of a Springer fibers $H_*(\mathcal{B}_x)$; this action descends to an action of $H(Z)$ on $H(\mathcal{B}_x)$. By the main result of the finite sets section, one might suspect that consideration of the modules $H(\mathcal{B}_x)$ leads to a classification of simple modules for $H(Z)$. This is true once we take into account the action of $G$ on $\mathcal{N}$.

Let $x \in \mathcal{N}$ be a nilpotent element and let $G(x)$ be the stabilizer of $x$ in $G$. Since $\mu$ is $G$-equivariant, there is an action of $G(x)$ on the Springer fiber $\mathcal{B}_x$, and hence on its homology $H(\mathcal{B}_x)$. The identity component $G(x)^\circ$ of $G(x)$ acts trivially on $H(\mathcal{B}_x)$, so we obtain an action of the component group $A(x) = G(x)/G(x)^\circ$ on $H(\mathcal{B}_x)$.

**Proposition 3.** *The actions of $A(x)$ and $H(Z)$ on $H(\mathcal{B}_x)$ commute.*

Since $A(x)$ is a finite group, we can decompose $H(\mathcal{B}_x)$ into a direct sum of isotypic subspaces for the (isomorphism classes of) irreducible representations of $A(x)$:

$$H(\mathcal{B}_x) = \bigoplus_{\phi \in \mathrm{Irrep}A(x)} H(\mathcal{B}_x)_\phi.$$

Here Irrep$A(x)$ denotes the finite set of isomorphism classes of irreducible representations of $A(x)$. Some of these isotypic subspaces may be zero; let $A(x)^\vee = \{\phi \in \mathrm{Irrep}A(x) \mid H(\mathcal{B}_x)_\phi \neq 0\}$. The preceding proposition implies that each isotypic subspace $H(\mathcal{B}_x)_\phi$ is preserved by the action of $H(Z)$.

Let $x, y \in \mathcal{N}$. If $x$ and $y$ are $G$-conjugate, i.e. $y = \mathrm{Ad}(g) \cdot x$ for some $g \in G$, then the action of $g$ gives an identification of:

- the component groups $A(x)$ and $A(y)$.
- the sets Irrep$A(x)$ and Irrep$A(y)$.
- the homology spaces $H(\mathcal{B}_x)$ and $H(\mathcal{B}_y)$.

If $\phi \in \mathrm{Irrep}A(x)$ corresponds to $\psi \in \mathrm{Irrep}A(y)$ under the second identification, then we say that the pairs $(x, \phi)$ and $(y, \psi)$ are $G$-conjugate. In this case, the isotypic subspaces $H(\mathcal{B}_x)_\phi$ and $H(\mathcal{B}_y)_\psi$ are in correspondence under the third identification, and are isomorphic as $H(Z)$-modules. The following theorem is the main result of this section:

**Theorem 4.** *The collection $\{H(\mathcal{B}_x)_\phi\}$, where $(x, \phi)$ runs over $G$-conjugacy classes of pairs $x \in \mathcal{N}$ and $\phi \in A(x)^\vee$, is a complete set of irreducible representations of $W$.*

Hence we obtain a geometric classification of the irreducible representations of any Weyl group. We invite the reader to compare this theorem with Proposition 1.

We make some remarks on the proof of this theorem. One ingredient is an ordering of the orbits of the diagonal action of $G$ on $\mathcal{B} \times \mathcal{B}$; this ordering gives rise to a filtration on $H(Z)$ and eventually one shows that $H(Z)$ is a direct sum of matrix algebras. Another approach is to develop sheaf-theoretic machinery and use a certain Fourier transform, as well as the powerful Beilinson-Bernstein-Deligne decomposition theorem.

**Remark** Consider the category $\mathsf{LS}^G(\mathbb{O}_x)$ of $G$-equivariant local systems (i.e. locally constant sheaves of finite-dimensional complex vector spaces) on the orbit $\mathbb{O}_x$ of $x$ in $\mathcal{N}$. This category has a fiber functor to the category of finite-dimensional complex vector spaces given by taking the stalk at $x$. Therefore, it is a neutral Tannakian category. Its Tannakian fundamental group is $A(x)$.

**Example.** Let $\mathfrak{g} = \mathfrak{sl}_n$. In this case, the Weyl group is the symmetric group $W = S_n$. It turns out that all component groups $A(x)$ act trivially, so the theorem gives the first of this series of bijections:

$$\mathrm{Irrep}(S_n) \longleftrightarrow \left\{ \begin{array}{c} G\text{-orbits} \\ \text{on } \mathcal{N} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes} \\ \text{of nilpotent matrices} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{partitions} \\ \text{of } n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy} \\ \text{classes in } S_n \end{array} \right\}.$$

The third bijection follows from the theory of Jordan normal forms.

# 4 Springer theory for $\mathfrak{sl}_n$

The aim of this section is to state a geometric classification of finite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_n$ using the pattern of describing modules for convolution algebras arising from fiber products.

The first step is to fix a positive integer $d$ bearing no relation to $n$. Let $\mathcal{F}$ denote the set of $n$-step partial flags of $\mathbb{C}^d$, that is, $\mathcal{F}$ consists of all sequences of subspaces

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^d$$

where the inclusions are not necessarily proper. Observe that the connected components of $\mathcal{F}$ are indexed by partitions of $d$ into $n$ integers. As with the flag variety $\mathcal{B}$, one can show that $\mathcal{F}$ has the natural structure of a smooth projective variety.

Let $N$ denote the set of linear maps $x \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^d)$ such that $x^n = 0$. Let

$$M = \{(x, F_\bullet) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1}\}.$$

One can identify $M$ with the cotangent bundle $T^*\mathcal{F}$; hence $M$ is a smooth variety. The natural map

$$\mu : M \to N,$$

that sends $(x, F_\bullet)$ to $x$, is proper and commutes with the action of $\mathrm{GL}_d(\mathbb{C})$. Let $\mathcal{F}_x$ denote the fiber $\mu^{-1}(x)$ of an element $x$ in $N$. The analogue of the Steinberg variety is the fiber product $Z = M \times_N M$. When we wish to emphasize the dependency on $d$, we write $Z_d$ instead of just $Z$. As in previous section, the Borel-Moore homology $H(Z)$ of $Z$ has a convolution algebra structure and $H(Z)$ is a subalgebra. Once again, we have an action of $H(Z)$ on $H(\mathcal{F}_x)$ for any $x \in N$. In this case, the component group acts trivially and two such representations $H(\mathcal{F}_x)$ and $H(\mathcal{F}_y)$ are isomorphic if and only if $x$ and $y$ are conjugate under the action of $\mathrm{GL}_d(\mathbb{C})$.

**Theorem 5.** *The collection $\{H(\mathcal{F}_x)\}$, where $x$ runs through $GL_d(\mathbb{C})$-conjugacy classes of $N$, is a complete collection of irreducible $H(Z)$-modules.*

The proof of this theorem is an adaptation of the proof of Theorem 4. The following result relates the Lie algebra $\mathfrak{sl}_n$ to the convolution algebras $H(Z) = H(Z_d)$ for various $d$:

**Theorem 6.** *For any positive integer $d$, there is a surjective algebra homomorphism $U(\mathfrak{g}) \to H(Z_d)$. Moreover, the action of $\mathfrak{sl}_n$ on any irreducible finite dimensional representation factors through $H(Z_d)$ for some $d$.*

The homomorphism can be defined explicitly by specifying the images of the Chevalley generators $\{e_i, f_i, h_i\}$ of $\mathfrak{sl}_n$ as linear combinations of fundamental classes in $H(Z_d)$. Some combinatorics of the $n$-step flag varieties is involved in making sure the correct relations hold. One can also appeal to sheaf-theoretic techniques. There is a way to interpret this result in the language of Nakajima's quiver varieties for a type A quiver; this approach can be generalized to arbitrary Kac-Moody algebras [2].

## 5 Epilogue: affine Hecke algebras and the Langlands program

Due to the time constraints of this talk and the amount of necessary background, we refrain from delving into the definition of the affine Hecke algebra. Instead, we note that the affine Hecke algebra of a complex semisimple group $G$ is an algebra over $\mathbb{Z}[q, q^{-1}]$ that encodes information about the Weyl group of $G$ and the weight lattice of $G$. Let $\mathbf{H}$ denote the affine Hecke algebra; in cases where we emphasize the dependency on the group $G$, we write $\mathbf{H}(G)$ instead.

It turns out that irreducible representations of the affine Hecke algebra can also be classified using geometric data, and the procedure follows a pattern similar to the one we have seen in the previous sections. This time, the convolution algebra that arises is the fiber product of certain fixed point varieties in the Springer resolution.

Let ${}^L G$ denote the Langlands dual of $G$. When we specialize $q$ to a prime $p$, the same data that classifies irreducible representations of $\mathbf{H}({}^L G)|_{q=p}$ also classifies so-called tamely unramified homomorphisms of the absolute Galois group of $\mathbb{Q}_p$ into the group ${}^L G$. On the other hand, work of Bernstein and Iwahori-Matsumoto shows that $\mathbf{H}({}^L G)|_{q=p}$ is isomorphic to the convolution algebra $\mathbb{C}[I \backslash G(\mathbb{Q}_p)/I]$ of complex-valued functions on a double coset space of $G(\mathbb{Q}_p)$ corresponding to the Iwahori subgroup $I$. These results establish the Langlands-Deligne conjecture, which is considered a starting point in the general Langlands program.

## References

[1] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*. Birkhäuser, 2000.

[2] V. Ginzburg. Lectures on Nakajima's quiver varieties, arxiv:0905.0686. 2009.